

# On proving syntactic properties of CPS programs

Olivier Danvy and Belmina Dzafic<sup>1</sup>

*BRICS*

*Department of Computer Science, University of Aarhus  
Building 540, Ny Munkegade, DK-8000 Aarhus C, Denmark.  
E-mail: {danvy,belmina}@brics.dk*

Frank Pfenning<sup>2</sup>

*School of Computer Science, Carnegie Mellon University  
5000 Forbes Ave., Pittsburgh, PA 15213-3891, USA.  
E-mail: fp@cs.cmu.edu*

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## Abstract

Higher-order program transformations raise new challenges for proving properties of their output, since they resist traditional, first-order proof techniques. In this work, we consider (1) the “one-pass” continuation-passing style (CPS) transformation, which is second-order, and (2) the occurrences of parameters of continuations in its output. To this end, we specify the one-pass CPS transformation relationally and we use the proof technique of logical relations.

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## 1 Introduction

We are interested in two syntactic properties of CPS programs: the occurrences of continuation identifiers and of parameters of continuations. The first occurrence formalizes a folklore property in the continuation community that “one  $k$  is enough” [10]. The second occurrence was informally stated in connection with the direct-style transformation, which is an inverse of the CPS transformation [3]. We address the second occurrence property here.

CPS programs are typically obtained by CPS transformation, and the canonical CPS transformation is due to Plotkin, in the mid-70’s [17]. It, however, gives rise to annoying “administrative reductions” that are interleaved with actual reductions. Proving properties of CPS programs such as relating

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their reduction steps with the corresponding reduction steps in direct style thus required Plotkin to develop a so-called “colon translation” [17] which has stuck [12,19].

In the late 80’s, however, a new CPS transformation was developed that operates in one pass and performs administrative reductions at transformation time [1,6,23]. This one-pass transformation is higher-order (or more precisely: second-order), and it is not clear how to prove properties about it, which is our goal here.

### **This work.**

We restate the one-pass CPS transformation in relational form and we present a proof technique using logical relations to prove a syntactic property of the output of the CPS transformation. This note grew out of a general study of the two syntactic properties mentioned above [5,7,9].

### **Overview.**

The rest of this note is organized as follows. In Section 2, we present a BNF of the  $\lambda$ -calculus in direct style, the corresponding BNF of the  $\lambda$ -calculus in CPS, and two successive refinements of the CPS transformation: Plotkin’s original specification, the one-pass specification in functional form, and our one-pass specification in relational form. In Section 3, we present the syntactic property of interest, and we prove that it is satisfied by the output of the CPS transformation. Section 4 concludes.

## **2 Direct style, continuation-passing style, and the CPS transformation**

### *2.1 Direct-style (DS) programs*

The BNF of the pure  $\lambda$ -calculus reads as follows. We refer to this  $\lambda$ -calculus as *direct style* (DS) to distinguish it from the *continuation-passing style* (CPS) calculus introduced below.

$r \in \text{DRoot}$	— DS terms	$r ::= e$
$e \in \text{DExp}$	— DS expressions	$e ::= e_0 e_1 \mid t$
$t \in \text{DTriv}$	— DS trivial expressions	$t ::= x \mid \lambda x.r$
$x \in \text{Ide}$	— identifiers	

The distinction between trivial expressions and (serious) expressions originates in Reynolds’s work [18].

### *2.2 Continuation-passing style (CPS) programs*

The BNF of CPS terms reads as follows. (NB: We distinguish between the original identifiers  $x$  coming from the direct-style term, and the fresh identifiers  $k$  and  $v$  denoting continuations and the arguments of continuations.)

$$\begin{aligned}
\llbracket e \rrbracket^{\text{DRoot}} &= \llbracket e \rrbracket^{\text{DExp}} \\
\llbracket e_0 e_1 \rrbracket^{\text{DExp}} &= \lambda k. \llbracket e_0 \rrbracket^{\text{DExp}} \lambda v_0. \llbracket e_1 \rrbracket^{\text{DExp}} \lambda v_1. v_0 v_1 k \\
&\quad \text{where } k, v_0 \text{ and } v_1 \text{ are fresh.} \\
\llbracket t \rrbracket^{\text{DExp}} &= \lambda k. k \llbracket t \rrbracket^{\text{DTriv}} \quad \text{where } k \text{ is fresh.} \\
\llbracket x \rrbracket^{\text{DTriv}} &= x \\
\llbracket \lambda x. r \rrbracket^{\text{DTriv}} &= \lambda x. \llbracket r \rrbracket^{\text{DRoot}}
\end{aligned}$$

Fig. 1. Plotkin's left-to-right, call-by-value CPS transformation

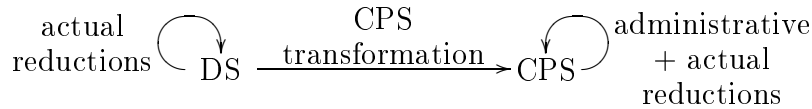
$r$	$\in \text{CRoot}$	— CPS terms	$r ::= \lambda k. e$
$e$	$\in \text{CExp}$	— CPS (serious) expression	$e ::= t_0 t_1 c \mid c t$
$t$	$\in \text{CTriv}$	— CPS trivial expression	$t ::= x \mid \lambda x. r \mid v$
$c$	$\in \text{CCont}$	— CPS continuations	$c ::= \lambda v. e \mid k$
$x$	$\in \text{Ide}$	— source identifiers	
$k$	$\in \text{Cont}$	— fresh continuation identifiers	
$v$	$\in \text{Var}$	— fresh parameters of continuations	

CPS terms are remarkable in that they satisfy the three properties of indifference, simulation, and translation [14,17,20]. Indifference: CPS terms are evaluation-order independent. Simulation: the CPS transformation encodes an evaluation order. Translation: there is an equational correspondence between direct-style and CPS calculi.

### 2.3 The CPS transformation

#### 2.3.1 From Plotkin's CPS transformation to the one-pass CPS transformation

Plotkin's original call-by-value CPS transformation is displayed in Figure 1, where it is phrased to match the syntactic domains of Section 2.1 [17]. Using it as a first-order rewriting system, however, gives rise to the notion of *administrative redexes*: redexes solely due to the CPS transformation and not corresponding to an actual reduction step in the original program. The corresponding administrative reductions are annoying because they are interleaved with actual reductions [12,13,17,19,20]:



Let us consider the following simple example, using Figure 1.

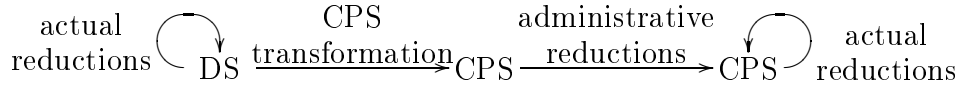
$$\llbracket \lambda x. x x \rrbracket^{\text{DRoot}} = \lambda k. k \lambda x. \lambda k. (\lambda k. k x) (\lambda v_0. (\lambda k. k x) (\lambda v_1. v_0 v_1 k))$$

$$\begin{aligned}
& \llbracket \cdot \rrbracket^{\text{DRoot}} : \text{DRoot} \rightarrow \text{CRoot} \\
& \llbracket e \rrbracket^{\text{DRoot}} = \lambda k. \llbracket e \rrbracket^{\text{DExp}} ([t] k t) \quad \text{where } k \text{ is fresh.} \\
\\
& \llbracket \cdot \rrbracket^{\text{DExp}} : \text{DExp} \rightarrow (\text{CTriv} \rightarrow \text{CExp}) \rightarrow \text{CExp} \\
& \llbracket e_0 e_1 \rrbracket^{\text{DExp}} = [\kappa] \llbracket e_0 \rrbracket^{\text{DExp}} [t_0] \llbracket e_1 \rrbracket^{\text{DExp}} [t_1] t_0 t_1 \lambda v. \kappa(v) \\
& \quad \text{where } v \text{ is fresh.} \\
& \llbracket t \rrbracket^{\text{DExp}} = [\kappa] \kappa(\llbracket t \rrbracket^{\text{DTriv}}) \\
\\
& \llbracket \cdot \rrbracket^{\text{DTriv}} : \text{DTriv} \rightarrow \text{CTriv} \\
& \llbracket x \rrbracket^{\text{DTriv}} = x \\
& \llbracket \lambda x. r \rrbracket^{\text{DTriv}} = \lambda x. \llbracket r \rrbracket^{\text{DRoot}}
\end{aligned}$$

Fig. 2. The one-pass CPS transformation formulated as a function

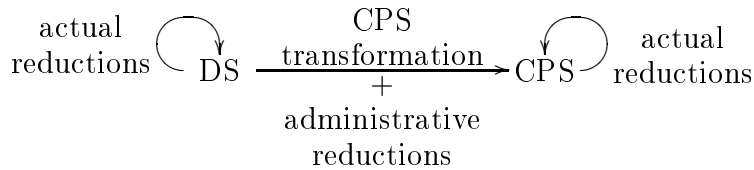
The CPS-transformed program contains two administrative redexes: the two occurrences of  $(\lambda k \dots) \dots$ . And reducing them yields two more administrative redexes.

It turns out, however, that administrative reductions can be factored out of a CPS program, so that the resulting reductions are actual ones:



Sabry and Felleisen have documented such an approach [2,19–21].

Furthermore, it turns out that CPS transformation and administrative reductions can be integrated into one, higher-order, rewriting system that directly produces a CPS program without administrative redexes, in one pass [1,6,8,23]:



Let us revisit the simple example above, using Figure 2.

$$\llbracket \lambda x. x x \rrbracket^{\text{DRoot}} = \lambda k. k \lambda x. \lambda k. x x k$$

We consider this higher-order CPS transformation here, as displayed in Figure 2, where it is phrased to match the syntactic domains of Sections 2.1 and 2.2. The one-pass CPS transformation requires meta-level abstractions, written as  $[t]e$ , and the corresponding applications, written as  $\kappa(t)$ , where  $\kappa$

$$\begin{array}{c}
\frac{\vdash e ; [t] k t \xrightarrow{\text{DExp}} e'}{\vdash e \xrightarrow{\text{DRoot}} \lambda k.e'} \qquad \frac{\vdash t \xrightarrow{\text{DTriv}} t'}{\vdash t ; \kappa \xrightarrow{\text{DExp}} \kappa(t')} \\
\\
\frac{\vdash e_1 ; [t_1] t_0 t_1 \lambda v.\kappa(v) \xrightarrow{\text{DExp}} e'_1 \quad \vdash e_0 ; [t_0] e'_1 \xrightarrow{\text{DExp}} e'}{\vdash e_0 e_1 ; \kappa \xrightarrow{\text{DExp}} e'} \\
\\
\frac{}{\vdash x \xrightarrow{\text{DTriv}} x} \qquad \frac{\vdash r \xrightarrow{\text{DRoot}} r'}{\vdash \lambda x.r \xrightarrow{\text{DTriv}} \lambda x.r'}
\end{array}$$

Fig. 3. The one-pass CPS transformation formulated as a judgment

ranges over meta-level functions from trivial CPS expressions to CPS expressions. The key type reads

$$\llbracket \cdot \rrbracket^{\text{DExp}} : \text{DExp} \rightarrow (\text{CTriv} \rightarrow \text{CExp}) \rightarrow \text{CExp}.$$

#### 2.4 The one-pass CPS transformation in relational form

For the purpose of our work here, Figure 3 re-expresses the one-pass CPS transformation of Figure 2 in relational form. It uses three judgments. A direct-style term  $r$  is transformed into a CPS term  $r'$  whenever the judgment

$$\vdash r \xrightarrow{\text{DRoot}} r'$$

is satisfied. Given a (higher-order) accumulator  $\kappa$ , a direct-style expression  $e$  is transformed into a CPS expression  $e'$  whenever the judgment

$$\vdash e ; \kappa \xrightarrow{\text{DExp}} e'$$

is satisfied. Finally, a direct-style trivial expression  $t$  is transformed into a CPS trivial expression  $t'$  whenever the judgment

$$\vdash t \xrightarrow{\text{DTriv}} t'$$

is satisfied.

These judgments can be interpreted operationally by assuming that  $r$ ,  $e$  and  $\kappa$ , or  $t$  are given and  $r'$ ,  $e'$ , and  $t'$  are to be constructed by building a derivation in a bottom-up fashion. The meta-level applications arising in two of the rules are reduced administratively.

NB: In the inference rule for applications,  $t_0$  is “new”, i.e., the deduction of the left premise is parametric in  $t_0$ . This parameter may, however, occur free in  $e'_1$ , which means that we can substitute an arbitrary trivial term  $t$  for  $t_0$  in this derivation and obtain a derivation of  $\vdash e_1 ; [t_1] t t_1 \lambda v.\kappa(v) \xrightarrow{\text{DExp}} e'_1[t/t_0]$ . This property is exploited crucially in the proof of Section 3.

$$\begin{array}{c}
\frac{\bullet \models_{\text{Var}}^{\text{CExp}} e}{\models_{\text{Var}}^{\text{CRoot}} \lambda k.e} \qquad \frac{\Xi \models_{\text{Var}}^{\text{CTriv}} t ; \Xi' \quad \Xi' \models_{\text{Var}}^{\text{CCont}} c}{\Xi \models_{\text{Var}}^{\text{CExp}} c t} \\
\\
\frac{\Xi \models_{\text{Var}}^{\text{CTriv}} t_1 ; \Xi_1 \quad \Xi_1 \models_{\text{Var}}^{\text{CTriv}} t_0 ; \Xi_0 \quad \Xi_0 \models_{\text{Var}}^{\text{CCont}} c}{\Xi \models_{\text{Var}}^{\text{CExp}} t_0 t_1 c} \\
\\
\frac{}{\Xi \models_{\text{Var}}^{\text{CTriv}} x ; \Xi} \qquad \frac{\models_{\text{Var}}^{\text{CRoot}} r}{\Xi \models_{\text{Var}}^{\text{CTriv}} \lambda x.r ; \Xi} \qquad \frac{}{\Xi, v \models_{\text{Var}}^{\text{CTriv}} v ; \Xi} \\
\\
\frac{}{\bullet \models_{\text{Var}}^{\text{CCont}} k} \qquad \frac{\Xi, v \models_{\text{Var}}^{\text{CExp}} e}{\Xi \models_{\text{Var}}^{\text{CCont}} \lambda v.e}
\end{array}$$

Fig. 4. Valid occurrences of parameters of continuations in a CPS term

### 2.5 Summary and conclusion

We have specified (1) the input language of Plotkin’s left-to-right, call-by-value CPS transformation, (2) the corresponding output language, which incidentally is closed under  $\beta$ -reduction, and (3) a one-pass version of Plotkin’s CPS transformation in relational form.

## 3 A syntactic property of CPS programs

The CPS transformation introduces two classes of fresh identifiers: the continuation identifiers  $k$  and the parameters of continuations  $v$ . We consider the occurrences of  $v$ ’s here.

Figure 4 characterizes the occurrence conditions on the formal parameters of continuations  $v$ , which occur in a stack-like fashion [3]. Here we use  $\Xi$  to range over stacks of continuation parameters defined below, where  $\bullet$  denotes the empty stack.

$$\Xi ::= \bullet \mid \Xi, v$$

Figure 4 should be read as follows. Given a CPS term  $\lambda k.e$ , the judgment

$$\models_{\text{Var}}^{\text{CRoot}} \lambda k.e$$

is satisfied whenever the parameters of continuations declared in  $e$  occur properly in  $e$ . Given a CPS expression  $e$  occurring in the scope of parameters of continuations properly listed in  $\Xi$ , the judgment

$$\Xi \models_{\text{Var}}^{\text{CExp}} e$$

is satisfied whenever the variables in  $\Xi$  and all the other parameters of continuations declared in  $e$  occur properly in  $e$ . Similarly, given a trivial term  $t$

occurring in the scope of parameters of continuations properly listed in  $\Xi$ , the judgment

$$\Xi \models_{\text{Var}}^{\text{CTriv}} t ; \Xi'$$

is satisfied whenever  $\Xi'$  is a prefix of  $\Xi$  and the remaining variables of  $\Xi$  occur properly in  $t$ . And finally, given a continuation  $c$  occurring in the scope of parameters of continuations properly listed in a list  $\Xi$ , the judgment

$$\Xi \models_{\text{Var}}^{\text{CCont}} c$$

is satisfied whenever all the parameters of continuations declared in  $c$  and the variables listed in  $\Xi$  occur properly in  $c$ .

This occurrence condition essentially says that formal parameters of continuations are introduced and used in a stack-like manner.

Let us prove that CPS-transforming a direct-style term  $r$  yields a CPS term  $r'$  whose continuation identifiers satisfy the occurrence conditions of Figure 4. In other words, we would like to show that

$$\text{if } \vdash r \xrightarrow{\text{DRoot}} r' \text{ then } \models_{\text{Var}}^{\text{CRoot}} r'.$$

Clearly, we cannot prove this inductively by itself since properties at the root of a term are defined in terms of the expressions it contains. The critical issue is the property of the higher-order accumulators  $\kappa$  we must prove (in the inductive conclusion) and require (in the inductive hypothesis) for the translation of expressions in Figure 3. In the CPS transformation, a higher-order accumulator is a (meta-level) function from trivial terms to expressions, which suggests the method of logical relations [22]. The idea behind binary logical relations is to consider two functions related if they map related arguments to related results. In unary form: a function is valid if it maps valid arguments to valid results. This kind of definition is pervasive in the application of logical frameworks to meta-theoretic reasoning (e.g., [9,15]). It works smoothly here.

Four notions of validity arise: for root terms, for serious expressions, for trivial expressions, and for accumulators. In their definitions, we must account for the context  $\Xi$  in which an expression might occur. For root terms, serious expressions, and trivial expressions, the notion of validity is derived directly from the property we are trying to prove; for accumulators it arises from the considerations of logical relations as motivated above. We also streamline the definitions by considering separately the case of a trivial variable  $v$ , since such a variable is never the result of the translation of a trivial direct-style term (see Theorem 3.2 (3)).

### Definition 3.1

- (1)  $r'$  is Var-valid if  $\models_{\text{Var}}^{\text{CRoot}} r'$ .
- (2)  $e'$  is  $\Xi$ -Var-valid if  $\Xi \models_{\text{Var}}^{\text{CExp}} e'$ .
- (3)  $t'$  is Var-valid if  $\Xi \models_{\text{Var}}^{\text{CTriv}} t'$  ;  $\Xi$  for every  $\Xi$ .
- (4)  $\kappa$  is  $\Xi$ -Var-valid if
  - (a)  $\Xi, v \models_{\text{Var}}^{\text{CExp}} \kappa(v)$ , and

- (b)  $\Xi \models_{\text{Var}}^{\text{CExp}} \kappa(t')$ , for any Var-valid  $t'$ .
- (5)  $c$  is  $\Xi$ -Var-valid if  $\Xi \models_{\text{Var}}^{\text{CCont}} c$ .

This definition is more complex than it may appear at first, since it involves meta-level applications  $\kappa(v)$  and  $\kappa(t')$  and therefore, implicitly, substitution.

**Theorem 3.2**

- (1) If  $\vdash r \xrightarrow{\text{DRoot}} r'$  then  $r'$  is Var-valid.
- (2) If  $\kappa$  is  $\Xi$ -Var-valid and  $\vdash e$ ;  $\kappa \xrightarrow{\text{DExp}} e'$  then  $e'$  is  $\Xi$ -Var-valid.
- (3) If  $\vdash t \xrightarrow{\text{DTriv}} t'$  then  $t'$  is Var-valid.

**Proof.** By mutual induction on the derivations  $\mathcal{R}$ ,  $\mathcal{E}$ , and  $\mathcal{T}$  of  $\vdash r \xrightarrow{\text{DRoot}} r'$ ,  $\vdash e$ ;  $\kappa \xrightarrow{\text{DExp}} e'$ , and  $\vdash t \xrightarrow{\text{DTriv}} t'$ , respectively.

In a slight abuse of notation, we write  $e(t_0)$  and  $\mathcal{E}(t_0)$  to indicate the dependence of  $e$  or  $\mathcal{E}$  on a parameter  $t_0$  and  $e(t)$  and  $\mathcal{E}(t)$  for the result of substituting  $t$  for  $t_0$  in  $e$  and  $\mathcal{E}$ , respectively.

$\mathcal{E}$

**Case  $\mathcal{R}$**   $= \frac{\vdash e; [t] k t \xrightarrow{\text{DExp}} e'}{\vdash e \xrightarrow{\text{DRoot}} \lambda k. e'}$

Then  $\kappa = [t] k t$  is  $\bullet$ -Var-valid:

- (a)  $\frac{\bullet, v \models_{\text{Var}}^{\text{CTriv}} v; \bullet}{\bullet, v \models_{\text{Var}}^{\text{CExp}} k v}$  holds, and
- (b)  $\frac{\bullet \models_{\text{Var}}^{\text{CTriv}} t'; \bullet}{\bullet \models_{\text{Var}}^{\text{CExp}} k t'}$  for any Var-valid  $t'$ .

Hence, by induction hypothesis (2) on  $\mathcal{E}$ ,  $\bullet \models_{\text{Var}}^{\text{CExp}} e'$ , and thus  $\models_{\text{Var}}^{\text{CRoot}} \lambda k. e'$ .

$\mathcal{E}_1(t_0)$   $\mathcal{E}_0$

**Case  $\mathcal{E}$**   $= \frac{\vdash e_1; [t_1] t_0 t_1 \lambda v. \kappa(v) \xrightarrow{\text{DExp}} e'_1(t_0) \quad \vdash e_0; [t_0] e'_1(t_0) \xrightarrow{\text{DExp}} e'}{\vdash e_0 e_1; \kappa \xrightarrow{\text{DExp}} e'}$

Assume  $\kappa$  is  $\Xi$ -Var-valid. We need to show that  $\kappa_0 = [t_0] e'_1(t_0)$  is  $\Xi$ -Var-valid, since then  $\Xi \models_{\text{Var}}^{\text{CExp}} e'$  by induction hypothesis (2) on  $\mathcal{E}_0$ . Thus we need to show Properties (a) and (b) of Definition 3.1(4) for  $\kappa_0$ .

- (a) We need  $\Xi, v_0 \models_{\text{Var}}^{\text{CExp}} \kappa_0(v_0)$ . Consider

$$\begin{array}{c} \mathcal{E}_1(v_0) \\ \vdash e_1; [t_1] v_0 t_1 \lambda v. \kappa(v) \xrightarrow{\text{DExp}} e'_1(v_0) \end{array}$$

We would like to show that

$$\kappa_1 = [t_1] v_0 t_1 \lambda v. \kappa(v)$$

is  $\Xi, v_0$ -Var-valid, since then  $e'_1(v_0) = \kappa_0(v_0)$  is  $\Xi, v_0$ -Var-valid by induction hypothesis (2) on  $\mathcal{E}_1(v_0)$ . Therefore we need to consider the two



cases of Definition 3.1(4).

- (a)  $\Xi, v_0, v_1 \models_{\text{Var}}^{\text{CExp}} \kappa_1(v_1)$ . We derive this as follows:

$$\frac{\overline{\Xi, v_0, v_1 \models_{\text{Var}}^{\text{CTriv}} v_1 ; \Xi, v_0} \quad \overline{\Xi, v_0 \models_{\text{Var}}^{\text{CTriv}} v_0 ; \Xi} \quad \begin{array}{c} \text{since } \kappa \text{ is } \Xi\text{-Var-valid} \\ \Xi, v \models_{\text{Var}}^{\text{CExp}} \kappa(v) \\ \hline \Xi \models_{\text{Var}}^{\text{CCont}} \lambda v. \kappa(v) \end{array}}{\Xi, v_0, v_1 \models_{\text{Var}}^{\text{CExp}} v_0 v_1 \lambda v. \kappa(v)}$$

- (b)  $\Xi, v_0 \models_{\text{Var}}^{\text{CExp}} \kappa_1(t'_1)$ , where  $t'_1$  is Var-valid. This is established by the derivation

$$\frac{\begin{array}{c} \text{since } t'_1 \text{ is Var-valid} \\ \Xi, v_0 \models_{\text{Var}}^{\text{CTriv}} t'_1 ; \Xi, v_0 \end{array} \quad \overline{\Xi, v_0 \models_{\text{Var}}^{\text{CTriv}} v_0 ; \Xi} \quad \begin{array}{c} \text{since } \kappa \text{ is } \Xi\text{-Var-valid} \\ \Xi, v \models_{\text{Var}}^{\text{CExp}} \kappa(v) \\ \hline \Xi \models_{\text{Var}}^{\text{CCont}} \lambda v. \kappa(v) \end{array}}{\Xi, v_0 \models_{\text{Var}}^{\text{CExp}} v_0 t'_1 \lambda v. \kappa(v)}$$

Thus  $\kappa_1$  is  $\Xi, v_0$ -Var-valid. Therefore, by induction hypothesis on  $\mathcal{E}_1(v_0)$ ,

$$\Xi, v_0 \models_{\text{Var}}^{\text{CExp}} \kappa_0(v_0).$$

- (b) We need  $\Xi \models_{\text{Var}}^{\text{CExp}} \kappa_0(t'_0)$  for any Var-valid  $t'_0$ . Consider

$$\begin{array}{c} \mathcal{E}_1(t'_0) \\ \vdash e_1 ; [t_1] t'_0 t_1 \lambda v. \kappa(v) \xrightarrow{\text{DExp}} \underbrace{e'_1(t'_0)}_{= \kappa_0(t'_0)} \end{array}$$

We would like to show that

$$\kappa_1 = [t_1] t'_0 t_1 \lambda v. \kappa(v)$$

is  $\Xi$ -Var-valid, so we can apply the induction hypothesis to  $\mathcal{E}_1(t'_0)$ .

Again, we need to consider the two clauses of Definition 3.1(4).

- (a)  $\Xi, v_1 \models_{\text{Var}}^{\text{CExp}} \kappa_1(v_1)$ . We derive this as follows:

$$\frac{\overline{\Xi, v_1 \models_{\text{Var}}^{\text{CTriv}} v_1 ; \Xi} \quad \begin{array}{c} \text{since } t'_0 \text{ is Var-valid} \\ \Xi \models_{\text{Var}}^{\text{CTriv}} t'_0 ; \Xi \end{array} \quad \begin{array}{c} \text{since } \kappa \text{ is } \Xi\text{-Var-valid} \\ \Xi, v \models_{\text{Var}}^{\text{CExp}} \kappa(v) \\ \hline \Xi \models_{\text{Var}}^{\text{CCont}} \lambda v. \kappa(v) \end{array}}{\Xi, v_1 \models_{\text{Var}}^{\text{CExp}} t'_0 v_1 \lambda v. \kappa(v)}$$

- (b)  $\Xi \models_{\text{Var}}^{\text{CExp}} \kappa_1(t'_1)$  for any Var-valid  $t'_1$ . We construct:

$$\frac{\begin{array}{c} \text{since } t'_1 \text{ is Var-valid} \\ \Xi \models_{\text{Var}}^{\text{CTriv}} t'_1 ; \Xi \end{array} \quad \begin{array}{c} \text{since } t'_0 \text{ is Var-valid} \\ \Xi \models_{\text{Var}}^{\text{CTriv}} t'_0 ; \Xi \end{array} \quad \begin{array}{c} \text{since } \kappa \text{ is } \Xi\text{-Var-valid} \\ \Xi, v \models_{\text{Var}}^{\text{CExp}} \kappa(v) \\ \hline \Xi \models_{\text{Var}}^{\text{CCont}} \lambda v. \kappa(v) \end{array}}{\Xi \models_{\text{Var}}^{\text{CExp}} t'_0 t'_1 \lambda v. \kappa(v)}$$

Hence  $\kappa_1$  is  $\Xi$ -Var-valid and thus  $\Xi \models_{\text{Var}}^{\text{CExp}} \underbrace{e'_1(t'_0)}_{= \kappa_0(t'_0)}$  by induction

hypothesis (2) on  $\mathcal{E}_1(t'_0)$ .

Thus  $\kappa_0$  is  $\Xi$ -Var-valid. Hence  $e'$  is  $\Xi$ -Var-valid by induction hypothesis (2) on  $\mathcal{E}_0$ .

$$\text{Case } \mathcal{E} = \frac{\frac{\mathcal{T}}{\vdash t \xrightarrow{\text{DTriv}} t'}}{\vdash t ; \kappa \xrightarrow{\text{DExp}} \kappa(t')}.$$

By induction hypothesis (3) on  $\mathcal{T}$ ,  $t'$  is Var-valid. Since we assume that  $\kappa$  is  $\Xi$ -Var-valid,  $\kappa(t')$  is also  $\Xi$ -Var-valid by clause 4b in Definition 3.1.

$$\text{Case } \mathcal{T} = \frac{\vdash x \xrightarrow{\text{DTriv}} x}{\Xi \models_{\text{Var}}^{\text{CTriv}} x ; \Xi} \text{ is an axiom for any } \Xi.$$

$$\text{Case } \mathcal{T} = \frac{\frac{\frac{\mathcal{R}}{\vdash r \xrightarrow{\text{DRoot}} r'}}{\vdash \lambda x.r \xrightarrow{\text{DTriv}} \lambda x.r'}}{\text{by i.h. (1) on } \mathcal{R}}$$

$$\text{We construct } \frac{\models_{\text{Var}}^{\text{CRoot}} r'}{\Xi \models_{\text{Var}}^{\text{CTriv}} \lambda x.r' ; \Xi}.$$

□

## 4 Conclusion

We have characterized an occurrence condition in CPS programs, Var-validity, and we have proven that this condition holds for the output of the one-pass CPS transformation. To this end, we developed a third-order proof technique matching the second-order nature of the one-pass CPS transformation.

Elsewhere [5,9], we investigate another, similar, occurrence condition on continuation identifiers. Using the same technique, we prove that the one-pass CPS transformation yields terms that satisfy this other occurrence condition. We then consider the closure of both occurrence conditions under  $\beta$ -reduction and their application to the direct-style transformation and to stack-based abstract machines for CPS programs.

We have also formalized most of the languages, transformations, properties, and proofs in Elf, a constraint logic-programming language based on the logical framework LF [9,11,16]. This formalization is small but non-trivial. It captures the computational content of the translations and the meta-theoretic reasoning in a declarative, yet executable way. Because Elf is built around the notions of substitution and meta-level function, the formalization is direct and (we find) elegant. It is also unusual in that since it abstracts over continuations, it requires third-order constants for the CPS transformation. This exemplifies a new technique for representing deductive systems in LF, which is interesting in its own right.

We can summarize this new technique as follows: we translate a two-level functional presentation to a relational representation in a logical framework by mapping “static” abstractions and applications directly to meta-level abstractions and applications. This means that static redexes of a two-level functional representation become  $\beta$ -redexes in the logical framework. Statically

convertible terms are therefore *definitionally equal*, avoiding explicit treatment of administrative reductions. A direct encoding of the meta-theory of such a representation will be third-order, since we reason about second-order objects.

Examples of two-level functional presentations include all one-pass CPS transformations, state-passing transformations, etc., and more generally the one-pass transformation into monadic style [13]. Type-directed partial evaluation provides another example of two-level functional presentations [4,9].

Finally, and most significantly, the encoding suggested the proof technique. This work thus demonstrates, on a small scale, the value of a logical framework as a conceptual tool in the theoretical study of programming languages.

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